Shifting Homomorphisms in Quandle Cohomology and Skeins of Cocycle Knot Invariants

J. Scott Carter
University of South Alabama
Mobile, AL 36688
carter@mathstat.usouthal.edu

Daniel Jelsovsky
University of South Florida
Tampa, FL 33620
jelsovsk@math.usf.edu

Seiichi Kamada
Osaka City University
Osaka 558-8585, JAPAN
kamada@sci.osaka-cu.ac.jp
skamada@mathstat.usouthal.edu

Masahico Saito
University of South Florida
Tampa, FL 33620
saito@math.usf.edu

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Abstract

Homomorphisms on quandle cohomology groups that raise the dimensions by one are studied in relation to the cocycle state-sum invariants of knots and knotted surfaces. Skein relations are also studied.

1 Introduction

A quandle is a set with a self-distributive binary operation (defined below) whose definition was motivated from knot theory. A (co)homology theory was defined in [2] for quandles, which is a modification of rack (co)homology defined in [10]. State-sum invariants using quandle cocycles as weights are defined [2] and computed for important families of classical knots and knotted surfaces [3]. Quandle homomorphisms and virtual knots are applied to this homology theory [4]. The invariants were applied to study knots, for example, in detecting non-invertible knotted surfaces [2].

In this paper, homomorphisms on quandle cohomology groups that raise the dimensions by one are studied in relation to the cocycle state-sum invariants of knots and knotted surfaces. Non-triviality of such homomorphisms is proved in Section 3. Skein relations are studied in Section 4 for some quandles. Preliminary material is contained in Section 2.

2 Definitions of Quandle (Co)Homology and Cocycle Invariants

2.1 Definition. A *quandle*, X, is a set with a binary operation $*$ such that
(I. IDEMPOTENCY) for any $a \in X$, $a * a = a$,
(II. RIGHT-INVERTIBILITY) for any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$, and
(III. SELF-DISTRIBUTIVITY) for any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [1],[8],[14],[18], and [19].

A map $f : X \to Y$ between two quandles (resp. racks) X, Y is called a quandle (resp. rack) homomorphism if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. A (quandle or rack) homomorphism is a (quandle or rack) isomorphism if it is bijective. An isomorphism between the same quandle (or rack) is an automorphism.

2.2 Examples. Any set X with the operation $x * y = x$ for any $x, y \in X$ is a quandle called the *trivial* quandle. The trivial quandle of n elements is denoted by T_n.

Any group G is a quandle by conjugation as operation: $a * b = b^{-1}ab$ for $a, b \in G$. Any subset of G that is closed under conjugation is also a quandle.

Let n be a positive integer. For elements $i, j \in \{0, 1, \ldots, n-1\}$, define $i * j = 2j - i$ where the sum on the right is reduced mod n. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n. This set can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation. We also represent the elements of R_3 by α, β, and γ, where the quandle multiplication is given by $x * y = z$ where $z \neq x, y$ when $x \neq y$ and $x * x = x$, for $x, y, z \in \{\alpha, \beta, \gamma\}$.

Any $\Lambda = \mathbf{Z}[T, T^{-1}]$-module M is a quandle with $a * b = Ta + (1-T)b$, $a, b \in M$, called an *Alexander quandle*. Furthermore for a positive integer n, a *mod-n Alexander quandle* $\mathbf{Z}_n[T, T^{-1}]/(h(T))$ is a quandle for a Laurent polynomial $h(T)$. The mod-n Alexander quandle is finite if the coefficients of the highest and lowest degree terms of h are ± 1.

See [1], [8], [14], or [19] for further examples.

2.3 Remark. Let X denote a quandle. From Axiom II, each element $b \in X$ defines a bijection $S(b) : X \to X$ with $aS(b) = a * b$. The bijection is an automorphism by Axiom III. For a word $w = b_1^{\epsilon_1} \ldots b_n^{\epsilon_n}$ where $b_1, \ldots, b_n \in X; \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$, we define $a * w = aS(w)$ by $aS(b_1)^{\epsilon_1} \ldots S(b_n)^{\epsilon_n}$. An automorphism of X is called an *inner-automorphism* of X if it is $S(w)$ for a word w. (The notation $S(b)$ follows Joyce's paper [14] and $a * w$ $(= a^w)$ follows Fenn-Rourke [8].)

Let $C_n^{\mathrm{R}}(X)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of a quandle X. Define a homomorphism $\partial_n : C_n^{\mathrm{R}}(X) \to C_{n-1}^{\mathrm{R}}(X)$ by

$$\partial_n(x_1, x_2, \ldots, x_n)$$

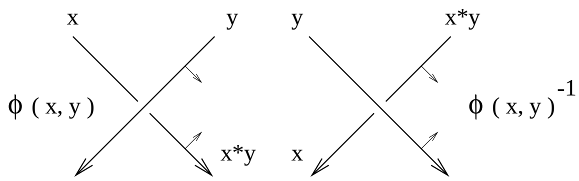

Figure 1: Crossings and weights

$$
\begin{aligned}
&= \sum_{i=2}^{n} (-1)^i \left[(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right. \\
&\left. - (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \right] \qquad (1)
\end{aligned}
$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C_*^{\rm R}(X) = \{C_n^{\rm R}(X), \partial_n\}$ is a chain complex.

Let $C_n^{\rm D}(X)$ be the subset of $C_n^{\rm R}(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$ if $n \geq 2$; otherwise let $C_n^{\rm D}(X) = 0$. If X is a quandle, then $\partial_n(C_n^{\rm D}(X)) \subset C_{n-1}^{\rm D}(X)$ and $C_*^{\rm D}(X) = \{C_n^{\rm D}(X), \partial_n\}$ is a sub-complex of $C_*^{\rm R}(X)$. Put $C_n^{\rm Q}(X) = C_n^{\rm R}(X)/C_n^{\rm D}(X)$ and $C_*^{\rm Q}(X) = \{C_n^{\rm Q}(X), \partial'_n\}$, where ∂'_n is the induced homomorphism. Henceforth, all boundary maps will be denoted by ∂_n.

For an abelian group G, define the chain and cochain complexes

$$
\begin{aligned}
C_*^{\rm W}(X; G) &= C_*^{\rm W}(X) \otimes G, & \partial &= \partial \otimes {\rm id}; & (2) \\
C^*_{\rm W}(X; G) &= {\rm Hom}(C_*^{\rm W}(X), G), & \delta &= {\rm Hom}(\partial, {\rm id}) & (3)
\end{aligned}
$$

in the usual way, where W $=$ D, R, Q.

2.4 Definition [2]. The nth *quandle homology group* and the nth *quandle cohomology group* [2] of a quandle X with coefficient group G are

$$H_n^{\rm Q}(X; G) = H_n(C_*^{\rm Q}(X; G)), \quad H^n_{\rm Q}(X; G) = H^n(C^*_{\rm Q}(X; G)). \qquad (4)$$

The cycle and boundary groups (resp. cocycle and coboundary groups) are denoted by $Z_n^{\rm Q}(X; G)$ and $B_n^{\rm Q}(X; G)$ (resp. $Z^n_{\rm Q}(X; G)$ and $B^n_{\rm Q}(X; G)$), so that

$$H_n^{\rm Q}(X; G) = Z_n^{\rm Q}(X; G)/B_n^{\rm Q}(X; G), \ H^n_{\rm Q}(X; G) = Z^n_{\rm Q}(X; G)/B^n_{\rm Q}(X; G).$$

We will omit the coefficient group G if $G = \mathbf{Z}$ as usual.

Assume that a finite quandle X is given. Pick a quandle 2-cocycle $\phi \in Z^2_{\rm Q}(X; G)$, and write the coefficient group, G, multiplicatively. Consider a crossing in the diagram. For each coloring of the diagram, evaluate the 2-cocycle on the quandle colors that appear near the crossing as described as follows: The first argument is the color on the under-arc away from which the normal to the over-arc points. The second argument is the color on the over-arc. Let τ denote a crossing, let $\epsilon(\tau)$ denote its sign, and let $\mathcal{C}$ denote a coloring.

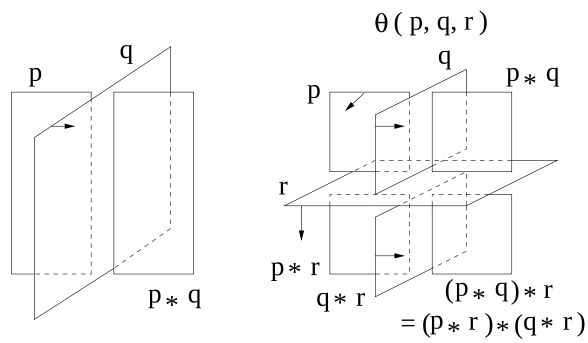

Figure 2: Colors at double curves and weights at triple points

2.5 Definition [2]. When the colors of the arcs are as describe above, the *(Boltzmann) weight of a crossing* is $B(\tau, \mathcal{C}) = \phi(x, y)^{\epsilon(\tau)}$.

The *state-sum*, is the expression

$$\Phi_\phi(K) = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}).$$

The product is taken over all crossings of the given diagram, and the sum is taken over all possible colorings. The values of the state-sums are taken to be in the group ring $\mathbf{Z}[G]$.

The coloring situation and the weights are depicted in Fig. 1. The state-sum invariant is similarly defined for knotted surfaces in 4-space using coloring conventions along double curves on projections (as in Fig. 2 left) and weights assigned to triple points (as in Fig. 2 right). In this case, signs $\epsilon = \pm 1$ are defined for triple points on projections, and the Boltzmann weight is defined by $\theta(p, q, r)^\epsilon$ using $\theta \in Z^3_{\rm Q}(X; G)$. See [2] for details.

It was proved in [2] that for classical knots and knotted surfaces in $\mathbf{R}^4$, the state-sums are knot invariants, by showing the invariance under Reidemeister moves and their 4-dimensional analogues (Roseman moves).

Furthermore, shadow colorings are defined using complementary regions, in addition to arcs and sheets, and are used to define state-sum invariants. Shadow colors are defined in [10] and used in [22]. The conventions and cocycle weights are depicted in Fig. 3. A shadow coloring is required to satisfy the condition depicted in the top left entry of Fig. 3. For $\theta \in Z^3_{\rm Q}(X; G)$, the value $\theta(q_0, q_1, q_2)$ corresponds to a crossing as depicted in top middle entry of Fig. 3. The Boltzmann weight is defined by $B(\tau, \mathcal{C}) = \theta(q_0, q_1, q_2)^{\epsilon(\tau)}$ for a shadow coloring $\mathcal{C}$, where $\epsilon(\tau)$ is the sign of the crossing τ. Then the state-sum is defined by $\Phi_\theta(K) = \sum_{\mathcal{C}} \prod_\tau B(\tau, \mathcal{C})$.

State-sum invariants using shadow colorings are similarly defined using $\xi \in Z^4_{\rm Q}(X; G)$, using the coloring convention depicted in Fig. 3 bottom left and middle, and using the correspondence to $\xi(q_0, q_1, q_2, q_3)$ depicted in Fig. 3 bottom right.

They are also invariants of classical knots and knotted surfaces. See [4, 5] for more details.

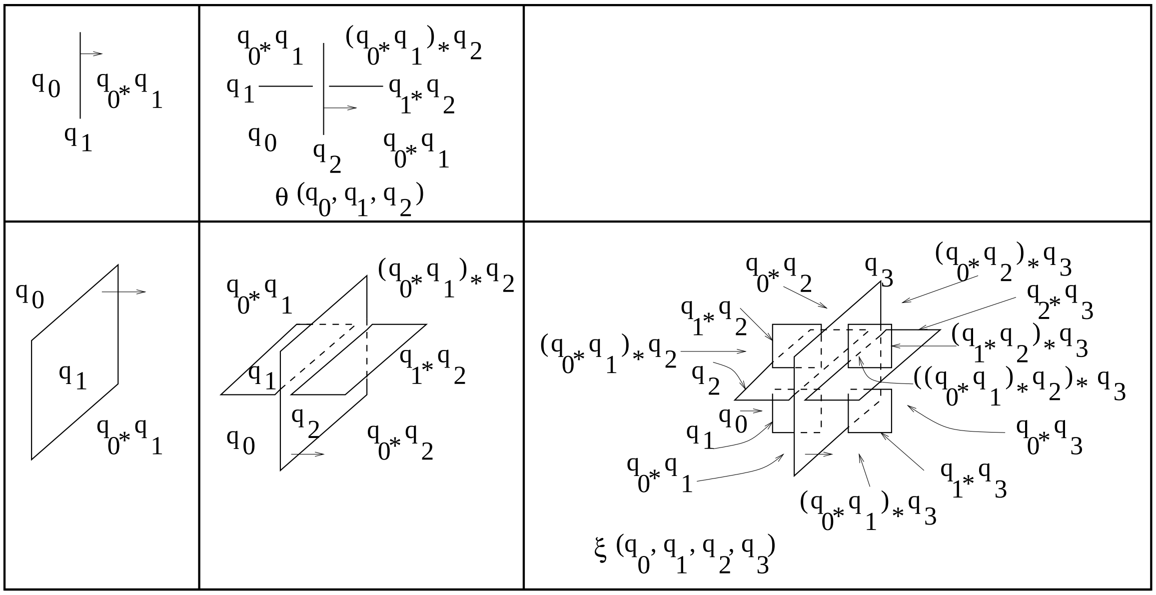

Figure 3: Shadow color conventions and weights at crossings and triple points

3 Shifting Homomorphisms

3.1 Definition. Let X be a quandle and G be a coefficient abelian group. Let $\rho = \rho^{(n)} : X^n \to X^{n-1}$ be the map defined by

$$\rho(x_1, x_2, \cdots, x_n) = \begin{cases} 0 & \text{if } \ x_1 = x_2, \\ (x_2, \cdots, x_n) & \text{if } \ x_1 \neq x_2. \end{cases}$$

Extending linearly we obtain $\rho_\sharp = \rho_\sharp^{(n)} : C_n^{\mathrm{R}}(X;G) \to C_{n-1}^{\mathrm{R}}(X;G)$. It is checked by computation that $\rho_\sharp(C_n^{\mathrm{D}}(X;G)) \subset C_{n-1}^{\mathrm{D}}(X;G)$, so that $\rho_\sharp$ induces $\rho_\sharp = \rho_\sharp^{(n)} : C_n^{\mathrm{Q}}(X;G) \to C_{n-1}^{\mathrm{Q}}(X;G)$ (the same notation $\rho_\sharp$ is used). Similarly define $\rho^\sharp = \rho^\sharp_{(n)} : C^n_{\mathrm{Q}}(X;G) \to C^{n+1}_{\mathrm{Q}}(X;G)$ by $(\rho^\sharp f)(x) = f(\rho_\sharp(x))$ for all $x \in C_{n+1}^{\mathrm{Q}}(X;G)$.

By computation we have

3.2 Lemma. *$\rho_\sharp \partial = -\partial \rho_\sharp$, and $\rho^\sharp \delta = -\delta \rho^\sharp$.*

Hence ρ induces homomorphisms on homology and cohomology groups, denoted by ρ_* and ρ^*. We call the homomorphisms $\rho_\#$, $\rho^\#$, ρ_*, ρ^* the *shifting homomorphisms.*

3.3 Proposition. *Let $\phi \in C^n_{\mathrm{Q}}(X;G)$ be an n-cochain for some quandle X and an abelian group G, and ρ be as above. Then $\rho^\sharp\phi$ is an $(n+1)$-cocycle (i.e. $\rho^\sharp\phi \in Z^{n+1}_{\mathrm{Q}}(X;G)$) if and only if ϕ is an n-cocycle ($\phi \in Z^n_{\mathrm{Q}}(X;G)$).*

Proof. We have that $\delta\rho^\sharp = -\rho^\sharp\delta$. So if $\phi \in Z^n_{\mathrm{Q}}(X;G)$, then

$$\delta\rho^\sharp\phi(x_0, \dots, x_n) = -\rho^\sharp\delta\phi(x_0, \dots, x_n) = 0.$$

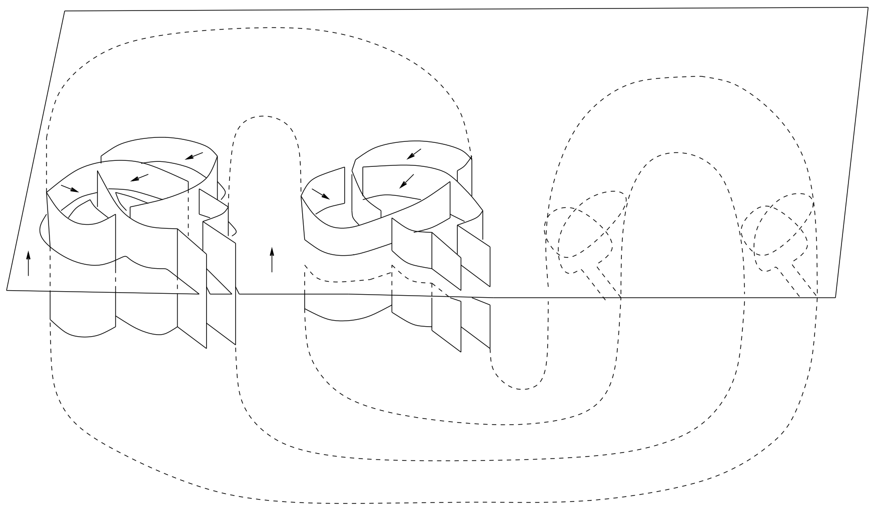

Figure 4: A diagram of 2-twist spun trefoil

If $\rho^\sharp\phi \in Z_{\rm Q}^{n+1}(X;G)$, then $\delta\rho^\sharp\phi(x_0,\dots,x_n) = 0$. On the other hand, $\delta\rho^\sharp\phi(x_0,\dots,x_n) = \delta\phi(x_1,\dots,x_n)$ if $x_0 \neq x_1$, so the result follows. $\square$

3.4 Remark. For some positive integer n, consider $\rho_\sharp : C_n^{\rm Q}(X) \to C_{n-1}^{\rm Q}(X)$. Clearly if $\eta \in Z_n^{\rm Q}(X)$ is a cycle, then $\partial_{n-1}\rho_\sharp\eta = -\rho_\sharp\partial_n\eta = 0$. However, if $\rho_\sharp(\zeta)$ is a cycle, the chain ζ need not be a cycle. For example, let $\zeta = (2,1,3) \in C_3(R_4)$, we see that $\rho(2,1,3) = (1,3)$ and $\partial_2(1,3) = 1 - 1*3 = 1 - 1 = 0$. However, $\partial_3(2,1,3) = (2,3)-(0,3)-(2,1)+(0,1) \neq 0$. Thus, $(2,1,3)$ is not a 3-cycle.

3.5 Proposition. $H^4_{\rm Q}(R_3;\mathbf{Z}_3) \neq 0$.

Proof. It was proved in [2] that if a coboundary is used to define the state-sum invariant, then the invariant is trivial (i.e., every state-sum term is 1). Analogous arguments apply to the state-sum invariants defined using shadow colorings to prove that if ξ is a coboundary, then the state-sum term is trivial (c.f. [5]). Hence we prove that there exists a cocycle $\xi \in Z^4_{\rm Q}(R_3;\mathbf{Z}_3)$ and a shadow coloring of a knotted surface diagram with a non-trivial state-sum term.

Specifically, let K be the 2-twist spun trefoil, and let $\phi \in Z^3_{\rm Q}(R_3;\mathbf{Z}_3)$ be a 3-cocycle defined (in [2]) by

$$\phi = \chi_{0,2,0} + \chi_{1,0,1} + \chi_{1,0,2} + \chi_{2,0,2} + \chi_{2,0,2} + \chi_{2,1,2} - \chi_{0,1,0} - \chi_{0,2,1},$$

where χ denotes the characterictic function

$$\chi_{a,b,c}(x,y,z) = \begin{cases} 1 & \text{if } (x,y,z) = (a,b,c), \\ 0 & \text{if } (x,y,z) \neq (a,b,c). \end{cases}$$

We show that there is a shadow coloring of a diagram of K which gives a non-trivial state-sum term with the cocycle $\xi = \rho^\sharp\phi \in Z^4_{\rm Q}(R_3;\mathbf{Z}_3)$.

For this purpose we use a diagram of K given in [23], as depicted in Fig. 4. In the figure, a diagram of trefoil with a small portion removed goes around the horizontal plane twice. The second time around (the right side of the figure) is depicted only schematically. The left-most trefoil goes over the plane with respect to the height direction, and the second goes under the plane, and this pattern is repeated one more time on the right side of the figure. The portion removed are connected to the plane by branch points, which are not depicted, as these portions of the diagram do not contain triple points and hence do not make any contribution to the state-sum invariant. The plane in fact represents a sphere. The continuous trace of the moving trefoil, together with the sphere attached to it via branch points, gives a diagram of 2-twist spun trefoil. The orientation normal vectors are also depicted in the figure. In [23] it was shown that with this diagram the state-sum is computed as

$$\begin{aligned}\Phi_\phi(K) = 3 + 2\ [\ &\phi(0,2,1)^{-1}\ \phi(2,1,0)\ \phi(0,1,2)^{-1}\ \phi(1,2,0) \\ &+\phi(1,0,2)^{-1}\ \phi(0,2,1)\ \phi(1,2,0)^{-1}\ \phi(2,0,1) \\ &+\phi(2,1,0)^{-1}\ \phi(1,0,2)\ \phi(2,0,1)^{-1}\ \phi(0,1,2)\ ].\end{aligned}$$

Each of these terms containing ϕ contributes t (a generator of $\mathbf{Z}_3$ in multiplicative notation), giving the state-sum invariant $3 + 6t$.

Consider the region R of the diagram which lies below the horizontal plane and is outside of the trace of the trefoil diagram. From the normal vectors, it is seen that the color of R appears in the first entry of a 4-cocycle ξ, when the state-sum invariant is computed using ξ and shadow colorings. Note also that given a coloring of the above diagram, any choice of color for R extends to a shadow coloring of the diagram (c.f. [5]). Hence we choose a shadow coloring with $2 \in R_3$ as the color assigned to R. With this choice, there are three terms in the above expression of ϕ that give the repetitive first and second entries for ξ: $\xi(2,2,1,0)$, $\xi(2,2,0,1)$, and $\xi(2,2,1,0)$. However, the corresponding triples evaluates trivially by ϕ. Therefore, this shadow coloring contributes t to the invariant. The result follows. $\square$

3.6 Remark. It is an interesting problem to determine when $\rho^{\sharp}_{(n)}$ is injective. The above proof illustrates the use of colored knot diagrams and shifting homomorphisms for solving this problem. In particular, colored knot diagrams are defined for higher dimensions [5], and the above method can be stated as follows as a conjecture.

Conjecture: Let X be a quandle, G an abelian group, $\phi \in Z^n_{\rm Q}(X;G)$, and K be an $(n-1)$-knot diagram in $\mathbf{R}^n$. If there is a color $\mathcal{C}$ of K with a non-trivial state-sum term for $\Phi_\phi(K)$, then $H^{n+1}_{\rm Q}(X;G) \neq 0$.

3.7 Proposition. *Let k and m be positive integers. Let X be the Alexander quandle $X = \mathbf{Z}_{mk}[T, T^{-1}]/(T - 1 \pm m)$. The orbit quandle* $\mathrm{Orb}(X)$ *is isomorphic to the trivial quandle T_m.*

Proof. We represent the elements of X as $\{0, 1, \dots, mk-1\}$ with quandle multiplication given by $a * b = (1 \mp m)a \pm mb \pmod{mk}$. Consider $f : X \to T_m$ given by $f(a) = a$ (mod m). Then

$$f(a * b) = (1 \mp m)a \pm mb = a = f(a) * f(b) \pmod{m}.$$

Thus $f : X \to T_m$ is a surjective quandle homomorphism which induces a surjective quandle homorphism $\tilde{f} : \mathrm{Orb}(X) \to T_m$. Suppose that $f(a) = f(b)$, so $b = a + ms$ for some integer s. Then $a * (a \pm s) = b$ (for $X = \mathbf{Z}_{mk}[T, T^{-1}]/(T - 1 \pm m)$ respectively), and a and b are in the same orbit. Thus $\tilde{f}$ is injective. $\square$

3.8 Corollary. $\mathrm{Orb}(\mathbf{Z}_8[T, T^{-1}]/(T - 3)) = T_2, \quad \mathrm{Orb}(\mathbf{Z}_8[T, T^{-1}]/(T - 5)) = T_4.$

Observe that either result can be obtained by direct computation since the quandles in question have 8 elements. In [4] it was proved that $\mathrm{rank}(H^n_{\mathrm{Q}}(X; G)) \geq \mathrm{rank}(H^n_{\mathrm{Q}}(\mathrm{Orb}(X); G))$ by considering the pull-backs of elements of $C^n_{\mathrm{Q}}(\mathrm{Orb}(X); G)$ in $C^n_{\mathrm{Q}}(X; G)$. Using the shifting homomorphism, we have the following.

3.9 Theorem. *For $X = \mathrm{Z}_8[T, T^{-1}]/(T - 5)$, there is a non-coboundary n-cocycle $\in Z^n_{\mathrm{Q}}(X; \mathbf{Z}_2)$ which is not the pull-back of an n-cycle $\in Z^n_{\mathrm{Q}}(T_4; \mathbf{Z}_2)$ for $n = 2, 3$. As a consequence,* $\mathrm{rank}(H^n_{\mathrm{Q}}(X; G)) > \mathrm{rank}(H^n_{\mathrm{Q}}(T_4; G))$.

Proof. For H^2, we used MAPLE to calculate that the value $\Phi_\theta(T(2, 4)) = 48 + 16t$ where the cocycle $\theta = \chi_{0,1} + \chi_{0,5} + \chi_{1,5} + \chi_{2,1} + \chi_{2,5} + \chi_{3,5} + \chi_{5,1} + \chi_{7,1}$ takes values in $\mathbf{Z}_2$, and $T(2, 4)$ denotes the $(2, 4)$-torus link.

Now, consider a cocycle $\phi \in Z^2_Q(X, \mathbf{Z}_2)$ which is the pull back of a 2-cocycle in T_4. Then

$$\phi = \sum_{i,j \in \{0,1,2,3\}, i \neq j} a_{i,j}(\chi_{i,j} + \chi_{i+4,j} + \chi_{i,j+4} + \chi_{i+4,j+4}),$$

where the $a_{i,j}$ s are constants. For any term $a_{i,j}(\chi_{i,j} + \chi_{i+4,j} + \chi_{i,j+4} + \chi_{i+4,j+4})$, consider its state-sum contribution for $T(2, 4)$ with the coefficient group $\mathbf{Z}_2$. We may consider $T(2, 4)$ as the closure of the braid σ_1^4. Then the only time any crossing will have non-trivial weight with respect to this particular $a_{i,j}(\chi_{i,j} + \chi_{i+4,j} + \chi_{i,j+4} + \chi_{i+4,j+4})$ is when the initial color vector of the braid is of the form (x, y) where $x \equiv i \pmod 4$ and $y \equiv j \pmod 4$. However, if this is the case, we have two crossings of this form. So the state-sum contribution of this color is still trivial (since $t^2 = 1$ in $\mathbf{Z}_2$). Thus, this part of the cocycle contributes 1 to the state-sum. Since the entire cocycle is made up of these parts, the state-sum of $T(2, 4)$ with $\mathbf{Z}_2$ is trivial, and, in fact, is 64. Since $64 \neq 48 + 16t$, θ and ϕ are not cohomologous. So, θ is not a pulled back cocycle, which means it is not accounted for by the rank of the trivial quandle.

For H^3, the state-sum invariant of K, using $\rho^\sharp\theta$ (where θ is as above) and using shadow colorings of $T(2, 4)$, is non-trivial. This can be seen using the shadow coloring depicted in Fig. 5.

If ϕ is a pullback of a 3-cocycle in $Z^3_Q(T_4; \mathbf{Z}_2)$, then it is of the form

$$\sum_{i,j,k \in 0,1,2,3, i \neq j, j \neq k} a_{i,j,k}(\chi_{i,j,k} + \chi_{i+4,j,k} + \chi_{i,j+4,k} + \chi_{i+4,j+4,k} + \chi_{i,j,k+4} + \chi_{i+4,j,k+4} + \chi_{i,j+4,k+4} + \chi_{i+4,j+4,k+4}).$$

For any term

$$a_{i,j,k}(\chi_{i,j,k} + \chi_{i+4,j,k} + \chi_{i,j+4,k} + \chi_{i+4,j+4,k} + \chi_{i,j,k+4} + \chi_{i+4,j,k+4} + \chi_{i,j+4,k+4} + \chi_{i+4,j+4,k+4})$$

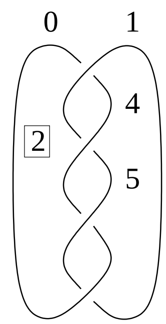

Figure 5: A shadow coloring of $T(2,4)$

consider its state-sum contribution for K over $\mathbf{Z}_2$. First note that for a given color the first term of every weight is the same. This is since the color on the region at infinity remains the same for all crossings of $T(2,4)$. Then, as with the 2-dimensional case, every choice of colors will either be trivial at every crossing, or cancel each other out. Thus, the state-sum invariant of K for ϕ is 512, and hence $\rho^\sharp\theta$ is not a pullback from the trivial quandle. $\square$

4 Skein Relations

Being state-sum invariants, the cocycle invariants have skein relations coming from the minimal polynomials of the corresponding R-matrices. However, the R-matrices are in general too large to compute by computers. In this section we give a method of finding skein relations using Burau matrices, which are much smaller than R-matrices. By skein relations, we mean linear formulas for the state-sum invariant, and we do not study the sufficiency of these as recursive formulas (i.e., whether or not they form a complete set of formulas to compute the invariant for any knots and links recursively).

The coefficient groups of cocycle groups used in this section are cyclic groups $G = \mathbf{Z}_p$ for some integer p, that are denoted multiplicatively, $\mathbf{Z}_p = \langle t | t^p = 1 \rangle = \{t^n | n = 0, 1, \cdots, p-1\}$. In this case, the state-sum takes values in $\mathbf{Z}[G] = \mathbf{Z}[t]/(t^p - 1)$.

An oriented n-tangle for a positive integer n is a tangle (diagram) with n strings going in at the top, and with n strings going out from the bottom of the diagram. Suppose that the colors on the top and bottom strings of a tangle are specified by vectors $[c_1, \cdots, c_n]$ and $[c'_1, \cdots, c'_n]$, respectively, such that the entries are elements of X. The state-sum for such a tangle is defined similarly, and denoted by

$$\Phi(T) \begin{bmatrix} c_1, & \cdots, & c_n \\ c'_1, & \cdots, & c'_n \end{bmatrix} = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}),$$

where $\mathcal{C}$ ranges over all colorings that restrict to the given colors, $[c_1, \cdots, c_n]$ and $[c'_1, \cdots, c'_n]$, on the boundary (top and bottom) segments, and τ ranges over all crossings of the tangle T.

4.1 Lemma. *Let T_i, $i = 1, \cdots, m$, be n-tangles for positive integers m, n. Let K_i, $i =$*

$1, \cdots, m$, *be classical knot or link diagrams such that they are all identical outside of a small ball inside which they have n-tangles T_i respectively. Suppose that*

- *the set of color vectors on the top and bottom strings of T_i are identical for all i, in the sense that if $[c_1, \cdots, c_n]$ and $[c'_1, \cdots, c'_n]$ are quandle vectors that color the top and bottom (respectively) strings of T_j for some j, $j = 1, \cdots, n$, then they color the strings of T_i for all $i = 1, \cdots, n$, uniquely.*

- *There exists a set of Laurent polynomials $f_i(t)$, $i = 1, \cdots, n$ such that for any top and bottom color vectors $[c_1, \cdots, c_n]$ and $[c'_1, \cdots, c'_n]$ that color T_is, the state-sum term $\Phi(T_i) = \Phi(T_i) \begin{bmatrix} c_1, & \cdots, & c_n \\ c'_1, & \cdots, & c'_n \end{bmatrix}$ for T_i satisfy the equality*

$$f_1(t)\Phi(T_1) + \cdots + f_n(t)\Phi(T_n) = 0.$$

Then the cocycle invariant satisfies the skein relation

$$f_1(t)\Phi(K_1) + \cdots + f_n(t)\Phi(K_n) = 0.$$

Proof. It follows from the state-sum definition (or by using the R-matrix description). □

4.2 Example. We illustrate the approach of finding skein relations using the above lemma and Burau matrices over R_4 (which is the Alexander quandle $\mathbf{Z}_2[T, T^{-1}]/(T^2 + 1)$) with the cocycle $\phi = \chi_{(0,1)}\chi_{(0,3)}$ (see [2]). Let T_+, T_0, and T_- be the 2-tangles represented by braid words σ_1^4, 1, and σ_1^{-4} respectively. Then in [2] it is observed that they satisfy the first condition of Lemma 4.1. For the color vectors $[i, i]$, $[2i, 2j]$, or $[2i + 1, 2j + 1]$ on top strings, for any i, j, the state-sum contribution is trivial ($1 \in G$) for all T_k. Thus the skein expression

$$f_+(t)\Phi(T_+) + f_0(t)\Phi(T_0) + f_-(t)\Phi(T_-) = 0$$

gives

$$f_+(t) + f_0(t) + f_-(t) = 0$$

for Laurent polynomials $f_k(t)$ that are to be determined. For other color vectors, $\Phi(T_+) = t$, $\Phi(T_0) = 1$, and $\Phi(T_-) = t^{-1}$. Hence we have

$$f_+(t)\ t + f_0(t) + f_-(t)\ t^{-1} = 0.$$

The choices $f_+(t) = 1 - t^{-1}$, $f_0 = t^{-1} + t$, and $f_-(t) = t - 1$ gives a solution, and we obtain a skein relation

$$(1 - t^{-1})\Phi(K_+) - (1 - t)\Phi(K_-) = (t - t^{-1})\Phi(K_0)$$

where K_k represent links with 2-tangles T_k in them.

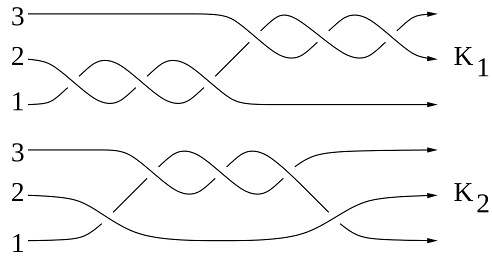

Figure 6: A skein relation for S_4

4.3 Example. In this example we examine skein relations for $S_4 = \mathbf{Z}_2[T, T^{-1}]/(T^2+T+1)$ and the cocycle $\phi = \prod_{x,y \neq T,\ x \neq y} \chi_{(x,y)}$. Here, $S_4 = \{0, 1, T, 1+T\}$, and the above product ranges over all pairs having no T, and χ denotes the characterictic function

$$\chi_a(b) = \begin{cases} t & \text{if } \ a = b \\ 1 & \text{if } \ a \neq b \end{cases}$$

for pairs a and b. Here the values in $G = \mathbf{Z}_2$ of cocycles are denoted multiplicatively.

First, recall from [3] that the trefoil and its mirror image has the same value $4(1+3t)$ with S_4 with the cocycle ϕ. Therefore S_4 has the skein relation

$$\Phi(K_{3+}) = \Phi(K_{3-})$$

where K_{3+} and K_{3-} denote the links with the braid word σ_1^3 and σ_1^{-3} in B_2 respectively (and the outside of these braid words are identical as usual with the skein relations).

Second, it is seen that the braid words depicted in Fig. 6 have the identity Burau representation with S_4 (as does σ_1^3). Let K_i, $i = 1, 2$ be links with the braids depicted in the figure, and let K_0 be the link with the identity braid word in place. Then we set up the skein relation

$$f_0(t)\Phi(K_0) + f_1(t)\Phi(K_1) + f_2(t)\Phi(K_2) = 0.$$

Give the numbers 1, 2, and 3 on the left arcs of Fig. 6 as depicted, from bottom to top. Let C_i be the colors on these arcs, for $i = 1, 2, 3$. We have the following cases.

Case (A): $C_1 = C_2 = C_3$ or C_i all distinct and $C_1 * C_2 = C_3$. In this case both K_1 and K_2 contribute 1 to the state-sum and hence gives a relation

$$f_0 + f_1 + f_2 = 0.$$

Case (B): $C_1 = C_2 \neq C_3$ or $C_1 \neq C_2 = C_3$. In this case both K_1 and K_2 contribute t to the state-sum and hence gives a relation

$$f_0 + t(f_1 + f_2) = 0.$$

Case (C): $C_1 = C_3 \neq C_2$ or C_i all distinct and $C_1 * C_2 \neq C_3$. In this case K_1 contributes 1 and K_2 contributes t. Therefore we obtain

$$f_0 + f_1 + tf_2 = 0.$$

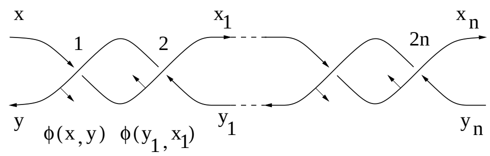

Figure 7: Antiparallel strings with $2n$ crossings

The three conditions reduce to $(t-1)f_i = 0$ for $i = 1, 2$ and $f_0 = -(f_1 + f_2)$. For example we obtain the relation

$$(t+1)(\Phi(K_1) + \Phi(K_2) - 2\Phi(K_3)) = 0.$$

The factor $(t+1)$ cannot be removed from this relation because it is not a unit.

Third, Maple computations show that with S_4 the Burau matrices coincide for the following variations of braid words (corresponding to the figure-eight knot), and the state-sum term also coincide for every color on the top strings. Hence the invariant does not change by replacement of one by another among the following: $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$, $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$, $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$, $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$.

4.4 Lemma. *Let P_n denote the two strings with $2n$ crossings and with oposite orientations as depicted in Fig. 7. (If n is positive, the crossings are all positive, and for negative n, the crossings are understood to be negative.)*

If the left two end points receive the colors x and y respectively in an Alexander quandle X, as depicted, then the right end points receive the colors x_n and y_n where

$$\begin{aligned} x_n &= (nT - (n-1))x + n(1-T)y, \\ y_n &= n(T-1)x + ((n+1) - nT)y. \end{aligned}$$

In particular, if the Alexander quandle has coefficients in $\mathbf{Z}_n$, then $x_n = x$ and $y_n = y$.

Furthermore, the crossings contribute the following terms to the state-sum expression if P_n is a part of a link:

$$\prod_{i=1}^{n} \phi(x_i, y_i)\phi(y_{i+1}, x_{i+1}).$$

Proof. Induction on n. □

4.5 Proposition. *If K and K' are related by a sequence of replacements of P_2 by $P_0 = I$ (the two anti-parallel strings with no crossing) or vice versa, then with the cocycle $\phi \in Z^2_{\rm Q}(S_4; \mathbf{Z}_2)$ in Example 4.3, $\Phi(K) = \Phi(K')$.*

Proof. It is computed using the preceding lemma that every color of P_2 contributes 1 to the state-sum. □

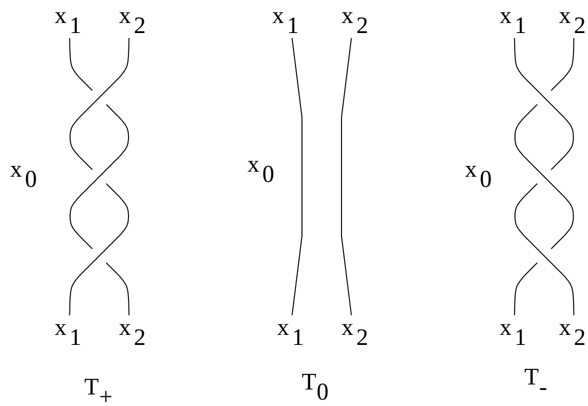

Figure 8: Tangles involved in the skein relation for R_3

State-sums with shadow colors have skein relation as well. Here we give an example with R_3. We label the elements of R_3 as $0, 1, 2$ with the quandle operation $i * j = 2j - i \pmod 3$.

Let $\xi \in Z^3_{\mathrm{Q}}(R_3; \mathbf{Z}_3)$ denote the 3-cocycle

$$\xi = \chi_{012}\chi_{021}\chi_{101}\chi_{201}\chi_{202}\chi_{102}$$

where

$$\chi_{abc}(x, y, z) = \begin{cases} t & \text{if } (x, y, z) = (a, b, c), \\ 1 & \text{if } (x, y, z) \neq (a, b, c). \end{cases}$$

Let Φ_ξ denote the state-sum invariant of a link associated with the cocycle ξ. Thus

$$\Phi_\xi = \sum_{\text{shadow colorings}} \prod_i \xi(a_i, b_i, c_i)^{\epsilon_i}$$

where (a_i, b_i, c_i) are the incoming colors at a crossing, ϵ_i is the sign of the crossing and the product ranges over all crossings. Consider the tangle T_+, T_0, and T_- that are depicted in Fig. 8. Let K_+, K_0, K_- be links with T_+, T_0, T_- in them, respectively.

4.6 Theorem. *In the notation above,*

$$(1 - t^{-1})\Phi_\xi(K_+) - (1 - t)\Phi_\xi(K_-) = (t - t^{-1})\Phi_\xi(K_0).$$

Proof. In the figure, let x_0, x_1, and x_2 denote colors by R_3 that are indicated in the figure. Thus x_0 is the color on the region to the left of the tangle, x_1 is the color on the top left string and x_1 is the color on the right string.

We find that the triple $f_+(t) = (1 - t^{-1})$, $f_0(t) = -(t - t^{-1})$, and $f_-(t) = t - 1$ is a solution of the equation

$$f_+(t)\Phi_\xi(T_+) + f_0(t)\Phi_\xi(T_0) + f_-(t)\Phi_\xi(T_-) = 0.$$

If $x_1 = 0$ and $x_2 = 1$, then the colors on the crossings of the tangles T_+, T_0 and T_- are read in the table below.

T_+	T_0	T_-
$(x_0, 0, 1)$	$\emptyset$	$(x_0, 2, 0)^{-1}$
$(x_0, 1, 2)$	$\emptyset$	$(x_0, 1, 2)^{-1}$
$(x_0, 2, 0)$	$\emptyset$	$(x_0, 0, 1)^{-1}$

For any value of x_0, the 3-cocycle ξ evaluates to t on T_+, 1 on T_0 and t^{-1} on T_-. Thus

$$tf_+(t) + f_0(t) + t^{-1}f_-(t) = 0.$$

When $x_1 \neq x_2$, then cocycle ξ evaluates similarly. When $x_1 = x_2$, we obtain, $f_+(t) + f_0(t) + f_-(t) = 0$. The result follows. □

4.7 Remark. Let Φ' denote the invariant associated with ξ^{-1}. Then

$$(1-t)\Phi'(K_+) - (1-t^{-1})\Phi'(K_-) = (t^{-1} - t)\Phi'(K_0).$$

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